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Unsteady state temperature fields in a slab induced by line sources

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Abstract

Solutions for the unsteady state temperature fields and heat fluxes from the surfaces due to single or multiple line sources in a slab and in a semi-infinite solid are derived by using integral transform techniques. The convective boundary condition (boundary condition of the third kind) is applied at the surfaces. Responses to a step change in the coil power are given. Also a cyclic case is studied, in which power in coils is alternately switched on and off as described by a cyclic square wave function. As an application the use of a floor or a wall as storage of heat from electric cables is discussed and equations applicable for dimensioning of the heat storage system and simulation of operation are given. The temperature isotherms generated by a line source are approximately circular in the vicinity of the line source. Based on this an approximate method to calculate heat losses in steady state from single or several pipes in a slab or ground with convective heat transfer at surface is illustrated.

Keywords: Line source; Transient heat conduction; Heat storage; Electric heating; Heat losses from pipes

1. Introduction

The study of point and line sources in heat conduction in solids dates back to Kelvin's theory in the 19th century [1]. The temperature field due continuous line source in an infinite solid has been applied as an approximation for the heating outside a buried electric cable [1], in the determination of thermal conductivity [1–5] and moisture content of soil [6,7]. Numerous methods to measure the thermal conductivity have been presented by applying heat sources. At present there exists standardized methods (EN, ISO, DIN, ASTM) for the determination of the thermal conductivity by the "hot-wire" method [5]. Thermal responses of a line source or a sink have been applied in studying heat extraction from soil using a heat pump [8–10]. Analogical physical problems applying point and line sources in fluid flow [11,12] or in the dispersion of pollutants [13] have been discussed. Solutions for temperature fields due to line source in different geometries have been presented (see e.g. [1,14,15]).

The steady state temperature profile in a solid around a heated pipe is close to one created by a line source. Approximate analytical solutions for the temperature distributions and heat transfer in steady state from a single pipe or several pipes in a slab or semi-infinite solid have been presented [16–20]. Periodic heat release from pipes in a slab has been studied numerically [21]. In the present paper transient temperature fields and heat losses due to line sources in slab geometry and in a semi-infinite solid are presented. A method is illustrated how this theory for line sources can be modified to estimate heat losses from pipes in a slab or ground.

The motivation for this study stems from a practical application, which is the use of a floor or a wall as storage of heat generated by electric coils. In storing electric heating the heating power can be switched off during daytime or other time when other consumption of electric energy is high. This makes it possible to direct the use of electric

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Nomenclature

a	$a = \lambda/(\rho c)$, thermal diffusivity, $m^2\,s^{-1}$
Bi	$Bi = G''R/\lambda_s$, effective Biot number
b	distance of cables from the bottom surface of the floor, m
c	specific heat capacity, $J\,kg^{-1}\,K^{-1}$
d	half distance between the line sources, m
Fo	$Fo = a_s t/R^2$, Fourier number, dimensionless time
G''	conductance/surface area, effective heat transfer coefficient, $W\,m^{-2}\,K^{-1}$
G'	conductance/pipe length, $W\,m^{-1}\,K^{-1}$
h	heat transfer coefficient, $W\,m^{-2}\,K^{-1}$
m	heat transfer resistance due to carpet on the floor, $m^2\,K\,W^{-1}$
$\dot{m}$	mass flow rate, $kg\,s^{-1}$
R	thickness of floor (without cover and insulation) for a slab, distance of the coil from the surface for a semi-infinite medium, m
R_p	outer radius of a pipe, m
r	distance from line source, m
s_b	thickness of the cover on the floor, m
T	temperature, K
T_{ref}	arbitrary reference temperature (but $T_{ref} \neq T_1$), K
t	time, s
x	space co-ordinate, m
y	space co-ordinate, m
z	space co-ordinate along fluid flow, m

Greek symbols

α	$\alpha = R/d$, ratio of lengths
β	$\beta = b/R$, ratio of lengths
δ	$\delta = R_p/R$, dimensionless pipe outer radius
ζ	$\zeta = y/d$, dimensionless space co-ordinate
η	$\eta = y/R$, dimensionless space co-ordinate
Θ	$\Theta = (T_s - T_1)\lambda_s/\phi'$, dimensionless temperature of solid
θ	$\theta = (T_s - T_1)/(T_{ref} - T_1)$, dimensionless temperature of solid
ϑ	$\vartheta = (T_2 - T_1)/(T_{ref} - T_1)$, dimensionless temperature of heated space 2
λ	thermal conductivity, $W\,m^{-1}\,K^{-1}$
ξ	$\xi = x/R$, dimensionless space co-ordinate
ρ	density, $kg\,m^{-3}$
Φ	$\Phi = \phi'/[\lambda_s(T_{ref} - T_1)]$, dimensionless power/length of line source
Φ^*	$\Phi^* = R\phi''/[\lambda_s(T_{ref} - T_1)]$, dimensionless heat flux
ϕ'	power /length of line source, $W\,m^{-1}$
ϕ''	heat flux, $W\,m^{-2}$
ϕ''_{max}	$\phi''_{max} = \phi'/(2d)$, average heat flux at surface 2 in stationary situation without heat losses at surface 1 (i.e. when $Bi_1 = 0$), $W\,m^{-2}\,K^{-1}$
Ψ	$\Psi = \lambda_s(T_2 - T_1)/\phi'$, dimensionless parameter

Subscripts

0	inlet
1	space under floor, side 1, power on period, space above a semi-infinite solid
2	heated room side, power off period
∞	stationary, final state
a	response to room temperature
av	average
b	cover on the solid
c	finite cosine transform
e	response to coil power
f	fluid
I	generalized transform
p	total period ($Fo_1 + Fo_2 = Fo_p$), pipe surface
s	solid

energy to off-peak hours when electric energy may be obtained at a cheaper price. The market prices of electric energy can momentarily rise even to 50 times the average during peaks, which occur mostly in cold winter days. Energy storage is useful with respect to the production of electric energy, since the peak in the production capacity can be dimensioned smaller or produced more efficiently compared to the case with direct electric heating. This heating method is in commercial use in Finland. Here the use of a floor as heat storage is considered as an application of the theory. Analytical solutions are derived for unsteady state temperature for a step change in the power/coil length and for the cyclic on/off-operation. Corresponding time-dependent heat releases to the room from the floor are also given. The results can be used in the dimensioning of the heat storage floor.

2. Thermal model

2.1. Model equations for a slab with line sources

The slab is illustrated in Fig. 1a. Due to symmetry it is possible to restrict the analysis to the area $y = 0 \ldots d$ in Fig. 1. Heat conduction and storage are described by Fourier's equation

$$\frac{1}{a_s}\frac{\partial T_s}{\partial t} = \frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2} + \frac{\phi'}{\lambda_s}\delta(x-b)\delta(y) \quad (1)$$

where the term on the left hand side describes heat storage, the two first terms on the right hand side describe two-dimensional heat conduction and the last term describes heat generation in a line source (or in an electric coil that

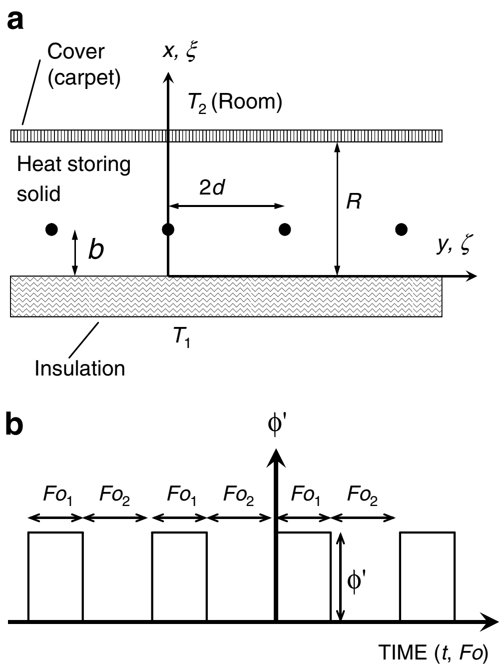

Fig. 1. Heat storing floor schematically (a) and coil power in cyclic operation (b).

is approximated as a line source) with power ϕ'. $\delta(x)$ is the Dirac delta function (see e.g. [22,23]) with properties $\delta(x-c)=0$, when $x \neq c$, $\delta(x-c)=\infty$, when $x=c$, and $\int_a^b \delta(x-c)F(x)\mathrm{d}x = F(c)$, when $a \leqslant c \leqslant b$. There is insulation with conductance G_1'' on the other side of the slab (or floor). There can be another room or ground on the side 1 of the slab (or floor). The boundary conditions on the surface of the floor, under the floor and due to symmetry are

$$G_2''(T_2 - T_{s,x=R}) = \lambda_s \left(\frac{\partial T_s}{\partial x}\right)_{x=R}$$
$$G_1''(T_1 - T_{s,x=0}) = -\lambda_s \left(\frac{\partial T_s}{\partial x}\right)_{x=0}$$
$$\left(\frac{\partial T_s}{\partial y}\right)_{y=0} = \left(\frac{\partial T_s}{\partial y}\right)_{y=d} = 0 \tag{2}$$

The thermal conductance of the surface of the floor includes the combined heat transfer coefficient by radiation and convection, h, and the thermal resistances of surface cover of the floor (s_b/λ_b) and of a possible carpet (m) on the floor, $G_2'' = 1/[1/h + s_b/\lambda_b + m]$.

In the general case the room temperature and the power of the cables can change with time and the initial temperature distribution in the floor is function of space co-ordinates. The temperature under the floor (or in space 1), T_1, is assumed constant. The solution could readily be generalised to the case where also temperature T_1 is changing with time. We use the following dimensionless variables: the Fourier number $Fo = a_s t/R^2$, dimensionless co-ordinates $\xi = x/R$ and $\zeta = y/d$, Biot numbers $Bi_1 = G_1''R/\lambda_s$ and $Bi_2 = G_2''R/\lambda_s$, length ratios $\alpha = R/d$ and $\beta = b/R$, dimensionless floor temperature $\theta = (T_s - T_1)/(T_{ref} - T_1)$, dimensionless room temperature $\vartheta = (T_2 - T_1)/(T_{ref} - T_1)$ and dimensionless power of line source $\Phi = \phi'/[\lambda_s(T_{ref} - T_1)]$. Eq. (1) and boundary conditions, Eq. (2), become in dimensionless form

$$\frac{\partial \theta}{\partial Fo} = \frac{\partial^2 \theta}{\partial \xi^2} + \alpha^2 \frac{\partial^2 \theta}{\partial \zeta^2} + \frac{1}{2}\alpha \Phi \delta(\xi - \beta)\delta(\zeta) \tag{3}$$

$$\left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=1} + Bi_2 \theta_{\xi=1} = Bi_2 \vartheta, \quad \left(\frac{\partial \theta}{\partial \xi}\right)_{\xi=0} - Bi_1 \theta_{\xi=0} = 0$$
$$\left(\frac{\partial \theta}{\partial \zeta}\right)_{\zeta=0} = \left(\frac{\partial \theta}{\partial \zeta}\right)_{\zeta=1} = 0 \tag{4}$$

The coefficient $\frac{1}{2}$ in Eq. (3) is due to symmetry, since only a half of the region is under consideration.

2.2. General solution

We apply the finite cosine transform [23]

$$\bar{f}(n) = \int_0^1 f(\zeta)\cos(n\pi\zeta)\,\mathrm{d}\zeta \tag{5}$$

which has the inverse transform

$$f(\zeta) = \bar{f}_c(n=0) + 2\sum_{n=1}^{\infty} \bar{f}_c(n)\cos(n\pi\zeta) \tag{6}$$

to Eq. (3), and we get

$$\frac{\partial \bar{\theta}_c}{\partial Fo} = \frac{\partial^2 \bar{\theta}_c}{\partial \xi^2} - \alpha^2 n^2 \pi^2 \bar{\theta}_c + \frac{\alpha}{2}\Phi\delta(\xi - \beta) \tag{7}$$

The integral transform [14,24]

$$f_1(\mu_k) = \int_0^1 K(\mu_k, \xi) f(\xi)\,\mathrm{d}\xi \tag{8}$$

where the kernel function is

$K(\mu_k, \xi) = [\mu_k \cos(\mu_k \xi) + Bi_1 \sin(\mu_k \xi)]/(\mu_k \sqrt{\gamma_k})$

and

$$\gamma_k = \frac{1}{2} + \frac{Bi_1^2}{2\mu_k^2} + \frac{1}{2}\left(1 + \frac{Bi_2}{Bi_1}\right)\left(\frac{1}{Bi_1} + \frac{Bi_2}{\mu_k^2}\right)\left(1 + \frac{Bi_1^2}{\mu_k^2}\right) \Bigg/ \left[\left(1 + \frac{Bi_2}{Bi_1}\right)^2 + \left(\frac{\mu_k}{Bi_1} - \frac{Bi_2}{\mu_k}\right)^2\right] \tag{9}$$

is applicable to problems in heat conduction in a plate with asymmetric convective boundary conditions. If $Bi_1 = 0$ i.e. perfect insulation below the floor, this transform is simplified to the generalized cosine transform [23]. The characteristic transcendental equation for the eigenvalues is

$$\tan \mu_k = \mu_k (Bi_1 + Bi_2)/(\mu_k^2 - Bi_1 Bi_2) \tag{10}$$

Suitable forms to solve the eigenvalues by iteration are

$$\mu_1 = \sqrt{\mu_1[\arctan(Bi_1/\mu_1) + \arctan(Bi_2/\mu_1)]} \tag{11}$$
$$\mu_k = (k-1)\pi + \arctan(Bi_1/\mu_k) + \arctan(Bi_2/\mu_k) \tag{12}$$

when $k = 1$ and $k > 1$, respectively. The inverse transform is [14,24]

$$f(\xi) = \sum_{k=1}^{\infty} K(\mu_k, \xi) f_1(\mu_k) \tag{13}$$

By applying transform, Eq. (8) to Eq. (7), we get

$$\frac{d\bar{\theta}_{cI}}{dFo} + (\alpha^2 n^2 \pi^2 + \mu_k^2)\bar{\theta}_{cI} = K(\mu_k, 1) Bi_2 \vartheta + \frac{1}{2}\alpha \Phi K(\mu_k, \beta) \tag{14}$$

The transformed floor temperature $\bar{\theta}_{cI}$ is obtained as the solution of differential equation (14). Then by applying the inverse transforms, Eqs. (6) and (13), we obtain the solution for the temperature distribution in the floor

$$\theta = \sum_{k=1}^{\infty} K(\mu_k, \xi)\left(\bar{\theta}_{cI}(n=0) + 2\sum_{n=1}^{\infty} \bar{\theta}_{cI} \cos(n\pi\zeta)\right) \tag{15}$$

2.3. Response to a step change in the coil power

We assume that the room temperature is constant and a constant power/length ϕ' in the coils is switched on. Initially the stationary temperature distribution in the floor, when there is no power on, is

$$\theta_{a,\infty} = \vartheta(Bi_2 + Bi_1 Bi_2 \xi)/(Bi_1 + Bi_2 + Bi_1 Bi_2) \tag{16}$$

The temperature distribution after the power is switched on is obtained from Eq. (15)

$$\theta = \theta_\infty - \frac{1}{2}\alpha\Phi \sum_{k=1}^{\infty} K(\mu_k, \beta) K(\mu_k, \xi) \times \left(\frac{1}{\mu_k^2} e^{-\mu_k^2 Fo} + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi\zeta)}{\alpha^2 n^2 \pi^2 + \mu_k^2} e^{-(\alpha^2 n^2 \pi^2 + \mu_k^2) Fo}\right) \tag{17}$$

$\theta_\infty = \theta_{a,\infty} + \theta_{e,\infty}$ is the stationary temperature distribution after the power has been on for long time. The contribution of the heat from the coils in the stationary temperature field is

$$\theta_{e,\infty} = \frac{1}{2}\alpha\Phi \sum_{k=1}^{\infty} K(\mu_k, \beta) K(\mu_k, \xi)\left(\frac{1}{\mu_k^2} + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi\zeta)}{\alpha^2 n^2 \pi^2 + \mu_k^2}\right) \tag{18}$$

which can be simplified. It can be shown by applying the residue theory that

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2 + b^2} = \frac{\pi}{4b}[\coth(\pi b) - 1]e^{bx} + \frac{\pi}{4b}[\coth(\pi b) + 1]e^{-bx} - \frac{1}{2b^2} \tag{19}$$

This gives an expression with one summation only

$$\theta_{e,\infty} = \frac{1}{2}\Phi \sum_{k=1}^{\infty} \frac{1}{\mu_k} K(\mu_k, \beta) K(\mu_k, \xi) \times [\coth(\mu_k/\alpha)\cosh(\mu_k \zeta/\alpha) - \sinh(\mu_k \zeta/\alpha)] \tag{20}$$

The average temperature of the floor surface 2 under the carpet can be obtained by integrating

$$\theta_{2,av} = \int_0^1 \theta(1, \zeta, Fo)\, d\zeta = \theta_{a,\infty}(1) + \int_0^1 \theta_e(1, \zeta, Fo)\, d\zeta$$

and we get

$$\theta_{2,av} = \frac{Bi_2 + Bi_1 Bi_2}{Bi_1 + Bi_2 + Bi_1 Bi_2} + \frac{1}{2}\alpha\Phi \sum_{k=1}^{\infty} K(\mu_k, \beta) K(\mu_k, 1) \times \frac{1}{\mu_k^2}(1 - e^{-\mu_k^2 Fo}) \tag{21}$$

The dimensionless heating power/surface area from the floor as function of time is obtained using the average surface temperature $\Phi_2 = Bi_2(\theta_{2,av} - \vartheta) = Bi_h(\theta_{p,av} - \vartheta) = \Phi_{2a,\infty} + \Phi_{2e}$,

$$\Phi_2^* = -\frac{Bi_1 Bi_2}{Bi_1 + Bi_2 + Bi_1 Bi_2}\vartheta + \frac{1}{2}\alpha Bi_2 \Phi \sum_{k=0}^{\infty} K(\mu_k, 1) K(\mu_{k,\beta}) \frac{1}{\mu_k^2}(1 - e^{-\mu_k^2 Fo}) \tag{22}$$

The first term is the contribution of the heat losses through the floor without coil power and the second is the contribution of the cables to the heating of the room. Correspondingly it is possible to find the average temperature of the other surface 1 (bottom) of the floor and the heat losses/surface area. The surface temperature distribution on the cover (felt by bare feet) is obtained from the temperature distribution at $\xi = 1$, $\theta_p = \theta(1, \zeta, Fo) + [\vartheta - \theta(1, \zeta, Fo)]G_2''/h$. If an additional carpet is used on the floor, its effect can be included in the conductance G_2''. The heat flux or heat loss at surface 1, Φ_1^*, is found in a similar way.

2.4. Intermittent heating

The power from the coils in the floor is shown in Fig. 1b in intermittent heating. Solutions for the case with arbitrary time dependent heat source power can be found by using Duhamel's method (see e.g. [14,24]). When the steady cyclic state has been reached, the solution can also be found by first solving the response to sinusoidal power and then presenting heat power as Fourier series and summing up the responses to each term. A third easier method in the present case to calculate the thermal response (temperature field and heat release from the floor) is obtained by presenting the coil power as square pulses and by using the superposition method. In cyclic on/off-heating the coil power changes according to square wave, which can be presented as a queue of positive and negative step changes. Then cyclic temperature distribution is achieved by the superposition of step responses i.e. summing up the contribution of positive and negative step response from the present to the past. If the step response is denoted by $f_s(Fo)$, then the response to the square wave heat source is $f(Fo) = f_s(Fo) - f_s(Fo + Fo_2) + f_s(Fo + Fo_p) - f_s(Fo + Fo_p + Fo_2) + \cdots$ for the power on period and

$f(Fo) = -f_s(Fo - Fo_1) + f_s(Fo) - f_s(Fo + Fo_2 +) \cdots$ for the power off period. It is assumed that the room temperature T_2 remains constant (due to auxiliary direct heating). By summing up to infinity in the past the temperature distribution in the floor during the period when the power is on, $0 \leqslant Fo \leqslant Fo_1$, is

$$\theta = \theta_{a,\infty} + \frac{\Phi}{2}\sum_{k=1}^{\infty}\frac{1}{\mu_k}K(\mu_k,\beta)K(\mu_k,\xi) \times \left[\coth\left(\frac{\mu_k}{\alpha}\right)\cosh\left(\frac{\mu_k\zeta}{\alpha}\right) - \sinh\left(\frac{\mu_k\zeta}{\alpha}\right)\right] - \frac{\Phi}{2}\alpha\sum_{k=1}^{\infty}K(\mu_k,\beta)K(\mu_k,\xi) \times \left(P_k\frac{e^{-\mu_k^2 Fo}}{\mu_k^2} + 2\sum_{n=1}^{\infty}Q_{n,k}\frac{e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo}}{\alpha^2n^2\pi^2+\mu_k^2}\cos(n\pi\zeta)\right) \tag{23}$$

where $P_k = [1 - e^{-\mu_k^2 Fo_2}]/[1 - e^{-\mu_k^2 Fo_p}]$ and $Q_{n,k} = [1 - e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo_2}]/[1 - e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo_p}]$. $\theta_{a,\infty}$ is defined by Eq. (16). During the period, when the power is switched off, $Fo_1 \leqslant Fo \leqslant Fo_p$, the temperature distribution is

$$\theta = \theta_{a,\infty} + \frac{\Phi}{2}\alpha\sum_{k=1}^{\infty}K(\mu_k,\beta)K(\mu_k,\xi) \times \left(L_k\frac{e^{-\mu_k^2 Fo}}{\mu_k^2} + 2\sum_{n=1}^{\infty}V_{n,k}\frac{e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo}}{\alpha^2n^2\pi^2+\mu_k^2}\cos(n\pi\zeta)\right) \tag{24}$$

where

$$L_k = e^{\mu_k^2 Fo_1} - [1 - e^{-\mu_k^2 Fo_2}]/[1 - e^{-\mu_k^2 Fo_p}]$$

and

$$V_{n,k} = e^{(\alpha^2n^2\pi^2+\mu_k^2)Fo_1} - [1 - e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo_2}]/[1 - e^{-(\alpha^2n^2\pi^2+\mu_k^2)Fo_p}]$$

The heat flux to the room in periodic operation is obtained in the same way as in the case of step response by calculating first the average surface temperature. The average heat flux when the power is on, $0 \leqslant Fo \leqslant Fo_1$, is

$$\phi_2''/\phi_{\max}'' = -\frac{2}{\alpha}\frac{Bi_1 Bi_2}{Bi_1 + Bi_2 + Bi_1 Bi_2}\Psi + \sum_{k=1}^{\infty}\frac{Bi_2}{\mu_k^2}K(\mu_k,\beta)K(\mu_k,1)\left(1 - P_k e^{-\mu_k^2 Fo}\right) \tag{25}$$

and when the power is off, $Fo_1 \leqslant Fo \leqslant Fo_p$,

$$\phi_2''/\phi_{\max}'' = -\frac{2}{\alpha}\frac{Bi_1 Bi_2}{Bi_1 + Bi_2 + Bi_1 Bi_2}\Psi + \sum_{k=1}^{\infty}\frac{Bi_2}{\mu_k^2}K(\mu_k,\beta)K(\mu_k,1)L_k e^{-\mu_k^2 Fo} \tag{26}$$

The heat losses as function of time under the floor in cyclic operation can be found correspondingly. If there is a constant base power on all the time in addition to the cyclic square wave, its stationary effect can be summed to the solutions.

The temperatures and heat fluxes at $Fo = 0$ and $Fo = Fo_p$ coincide. The calculations of heat fluxes at $\xi = 0$ and $\xi = 1$ in stationary situation can be performed faster using the relations

$$\sum_{k=1}^{\infty}\frac{Bi_1}{\mu_k^2}K(\mu_k,\beta)K(\mu_k,0) = \frac{\sigma}{1+\sigma}$$
$$\sum_{k=1}^{\infty}\frac{Bi_2}{\mu_k^2}K(\mu_k,\beta)K(\mu_k,1) = \frac{1}{1+\sigma} \tag{27}$$

where $\sigma = (Bi_1/Bi_2)[1 + (1-\beta)Bi_2]/[1 + \beta Bi_1]$. These relations were deduced from stationary energy balance: the heat released from coils is equal sum of heat released from the surfaces i.e. $\phi_{\max}'' = \phi'/(2d) = \phi_{1\infty}'' + \phi_{2\infty}''$, when $T_1 = T_2$. In addition the proportion of the heat release from surface 1 to that from surface 2 is proportional to total conductances from plane $\xi = \beta$ to spaces 1 and 2 i.e. $\phi_{1\infty}''/\phi_{2\infty}'' = \sigma$. The validity of Eq. (27) was also tested by calculations. Due to the exponential term $\exp(-\mu_k^2 Fo)$ the convergence of the calculations with the time dependent solutions is quite rapid and only few first terms in infinite summations are required except when Fo is very small.

2.5. Step response of a single coil in the floor

We consider a single coil in the floor (in Fig. 1 $d \to \infty$) as the limit. We apply a new dimensionless co-ordinate $\eta = y/R$ and Eq. (3) becomes

$$\frac{\partial\theta}{\partial Fo} = \frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}\Phi\delta(\xi - \beta)\delta(\eta) \tag{28}$$

Boundary conditions except latter symmetry condition in Eq. (4) are valid. Then, instead of the finite cosine transform, Eq. (5), we apply the Fourier cosine transform on the half line [23] and its inverse transform

$$\bar{f}_c(\omega) = \int_0^{\infty} f(\eta)\cos(\omega\eta)\,d\eta$$
$$f(\eta) = \frac{2}{\pi}\int_0^{\infty}\bar{f}_c(\omega)\cos(\omega\eta)\,d\omega \tag{29}$$

respectively. Then the solution for the temperature field becomes

$$\theta = \theta_{a\infty} + \frac{\Phi}{\pi}\sum_{k=1}^{\infty}K(\mu_k,\xi)K(\mu_k,\beta) \times \int_0^{\infty}\frac{\cos(\omega\eta)}{\omega^2+\mu_k^2}[1 - e^{-(\omega^2+\mu_k^2)Fo}]\,d\omega \tag{30}$$

The integral in Eq. (30) can be evaluated using tables [25] and Eq. (30) is simplified into

$$\theta = \theta_{a\infty} + \frac{\Phi}{2}\sum_{k=1}^{\infty}\frac{1}{\mu_k}K(\mu_k,\xi)K(\mu_k,\beta)\left[e^{-\mu_k|\eta|} - \cosh(\mu_k\eta) + \frac{1}{2}e^{-\mu_k\eta}\mathrm{erf}\left(\mu_k\sqrt{Fo} - \frac{\eta}{2\sqrt{Fo}}\right) + \frac{1}{2}e^{\mu_k\eta}\mathrm{erf}\left(\mu_k\sqrt{Fo} + \frac{\eta}{2\sqrt{Fo}}\right)\right] \tag{31}$$

where erf is the error function [26]. In stationary situation as the limit when $Fo \to \infty$, $\theta_\infty = \theta_{a\infty} + \theta_{e\infty}$, where the contribution of the line source is

$$\theta_{e\infty} = \frac{\Phi}{2}\sum_{k=1}^{\infty}\frac{e^{-\mu_k|\eta|}}{\mu_k}K(\mu_k,\xi)K(\mu_k,\beta) \tag{32}$$

2.6. Response of line sources in semi-infinite solid

We consider the temperature field due to single or several constant line sources in a semi-infinite solid medium or ground (Fig. 2), which corresponds to the limit $R \to \infty$. The heat transfer is described by Eq. (1), but the co-ordinate system is little different (see Fig. 2). The boundary condition at the surface is

$$G''(T_{s,x=0} - T_2) = \lambda_s\left(\frac{\partial T_s}{\partial x}\right)_{x=0} \tag{33}$$

and the initial temperature of the solid is constant and equal to ambient temperature. At time $t = 0$ power ϕ' of a line source is switched on. We choose the dimensionless variables $Fo = a_s t/R^2$, $\xi = x/R$, $\eta = y/R$ for a single line source (Fig. 2a) and $\zeta = y/d$ for infinite queue of line sources (Fig. 2b), Biot number $Bi = G''R/\lambda_s$ and dimensionless temperature $\Theta = (T_s - T_1)\lambda_s/\phi'$. Eq. (1) and boundary conditions become for a single line source

$$\frac{\partial\Theta}{\partial Fo} = \frac{\partial^2\Theta}{\partial\xi^2} + \frac{\partial^2\Theta}{\partial\eta^2} + \delta(\xi-1)\delta(\eta),$$
$$Bi\Theta_{\xi=0} = \left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi=0}, \quad \left(\frac{\partial\Theta}{\partial\eta}\right)_{\eta=0} = 0 \tag{34}$$

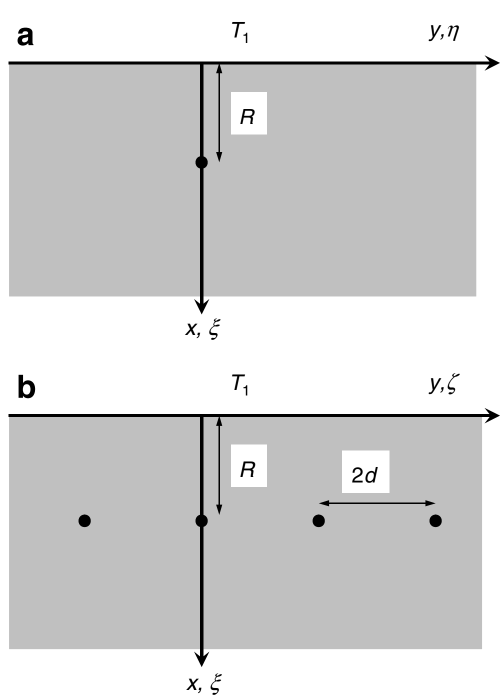

Fig. 2. Semi-infinite solid with single (a) or several (b, infinite queue) line sources.

The coefficient $\frac{1}{2}$ is added in the heat source term, when only a half region is under consideration. Initially T_s $(t=0) = T_1$ i.e. $\Theta(Fo=0) = 0$, but in case $\Theta(Fo=0) \neq 0$ its effect can be superposed to the solution. The solution for the temperature field due to single line source can be found by applying the cosine transform, Eq. (29) and the modified Fourier transform and its inverse transform [23]

$$\bar{f}_m(\gamma) = \int_0^\infty f(\xi)g(\gamma,\xi)\,d\xi, \quad f(\xi) = \frac{2}{\pi}\int_0^\infty \bar{f}_m(\gamma)\frac{g(\gamma,\xi)}{Bi^2+\gamma^2}\,d\gamma \tag{35}$$

where $g(\gamma,\xi) = \gamma\cos(\gamma\xi) + Bi\sin(\gamma\xi)$. The temperature field becomes

$$\Theta = \frac{1}{\pi}\int_0^\infty\left\{e^{-\eta\gamma} - \cosh(\eta\gamma) + \frac{1}{2}e^{-\eta\gamma}\mathrm{erf}\left(\gamma\sqrt{Fo} - \frac{\eta}{2\sqrt{Fo}}\right) + \frac{1}{2}e^{\eta\gamma}\mathrm{erf}\left(\gamma\sqrt{Fo} + \frac{\eta}{2\sqrt{Fo}}\right)\right\}\frac{g(\gamma,1)g(\gamma,\xi)}{\gamma(Bi^2+\gamma^2)}\,d\gamma \tag{36}$$

The steady state temperature distribution ($Fo \to \infty$) is

$$\Theta = \frac{1}{\pi}\int_0^\infty \frac{g(\gamma,1)g(\gamma,\xi)e^{-\eta\gamma}}{\gamma(Bi^2+\gamma^2)}\,d\gamma \tag{37}$$

This gives the maximum temperature at the surface above the line source at the location $\xi = \eta = 0$, $\Theta(0,0) = e^{Bi}E_1(Bi)/\pi$, at steady state. E_1 is the exponential integral [26]. In the case of high heat transfer coefficient at the surface the solution for the transient case can readily be found as the limit $Bi = \infty$ from Eq. (36). However, a more convenient solution can be found by applying the method of images [1]. When $Bi = \infty$, the boundary condition at the surface becomes $\Theta(\xi = 0, \eta) = 0$, which can be created by summing the effects of a positive and a negative line source, which are situated symmetrically with respect to the surface level $\xi = 0$ in an infinite solid. This creates a zero isotherm at the level $x = 0$, which is the correct boundary condition for the semi-infinite solid under study with the positive line source. The solution for a single line source (in an infinite solid) is [1]

$$\Theta = \frac{1}{4\pi}E_1\left(\frac{r^2}{4at}\right), \quad E_1(z) = \int_1^\infty \frac{e^{-tz}}{t}\,dt \tag{38}$$

where r is the distance from the line source. Then the solution for the temperature distribution in the semi-infinite solid is [1]

$$\Theta = \frac{1}{4\pi}\left\{E_1\left(\frac{(\xi-1)^2+\eta^2}{4Fo}\right) - E_1\left(\frac{(\xi+1)^2+\eta^2}{4Fo}\right)\right\} \tag{39}$$

The heat flux distribution at the surface $\xi = 0$ is obtained as

$$\phi'' = \frac{\phi'}{\pi R}\frac{1}{\eta^2+1}e^{-(\eta^2+1)/(4Fo)} \tag{40}$$

and the heat flow/length at the surface is $\phi'(\xi=0)/\phi' = 1 - \mathrm{erf}[1/(2\sqrt{Fo})]$. The well-known steady state ($Fo \to \infty$) temperature distribution is

$$\Theta = \frac{1}{4\pi} \ln\left(\frac{(\xi+1)^2+\eta^2}{(\xi-1)^2+\eta^2}\right) \tag{41}$$

which has circular isotherms making it applicable to estimate heat losses from pipe buried in the ground.

In the case with several line sources (Fig. 2b) there is symmetry condition at $\zeta = 1$ in addition to one at $\zeta = 0$. We apply the finite cosine transform, Eqs. (5) and (6), and the modified Fourier transform, Eq. (35). The heat transfer is described by Eq. (3) with $\beta = 1$. The solution for the temperature field is

$$\Theta = \frac{\alpha}{\pi}\int_0^\infty \left\{\frac{1}{\gamma^2}\left(1-e^{-\gamma^2 Fo}\right) + 2\sum_{n=1}^{\infty}\frac{\cos(n\pi\zeta)}{(\alpha n\pi)^2+\gamma^2} \times \left(1-e^{-\{(\alpha n\pi)^2+\gamma^2\}Fo}\right)\right\}\frac{g(\gamma,1)g(\gamma,\xi)}{Bi^2+\gamma^2}\,d\gamma \tag{42}$$

which becomes in steady state ($Fo \to \infty$)

$$\Theta = \frac{1}{2\pi}\int_0^\infty \left\{[\coth(\gamma/\alpha)-1]e^{\gamma\zeta/\alpha} + [\coth(\gamma/\alpha)+1]e^{-\gamma\zeta/\alpha}\right\} \times \frac{g(\gamma,1)g(\gamma,\xi)}{\gamma(Bi^2+\gamma^2)}\,d\gamma \tag{43}$$

In the special case with low thermal resistance at the surface ($Bi \to \infty$), the temperature field can be found in a similar way as before by summing up the contributions of positive and negative line sources,

$$\begin{aligned}\Theta = \frac{1}{4\pi}\sum_{n=0}^{\infty}\Bigg\{ & E_1\left(\frac{(\xi-1)^2+(\zeta+2n)^2/\alpha^2}{4Fo}\right) \\ & -E_1\left(\frac{(\xi+1)^2+(\zeta+2n)^2/\alpha^2}{4Fo}\right) \\ & +E_1\left(\frac{(\xi-1)^2+[\zeta-2(n+1)]^2/\alpha^2}{4Fo}\right) \\ & -E_1\left(\frac{(\xi+1)^2+[\zeta-2(n+1)]^2/\alpha^2}{4Fo}\right)\Bigg\}\end{aligned} \tag{44}$$

and the heat flux at the surface is

$$\phi'' = \frac{\phi'}{4R}\sum_{n=0}^{\infty}\left\{\frac{\exp\{-[(\zeta+2n)^2/\alpha^2+1]/(4Fo)\}}{(\zeta+2n)^2/\alpha^2+1} + \frac{\exp\{-[(\zeta-2(n+1))^2/\alpha^2+1]/(4Fo)\}}{[\zeta-2(n+1)]^2/\alpha^2+1}\right\} \tag{45}$$

The steady state temperature distribution is obtained as the limit ($Fo \to \infty$)

$$\Theta = \frac{1}{4\pi}\sum_{n=0}^{\infty}\ln \times \left(\frac{\{(\xi+1)^2+(\zeta+2n)^2/\alpha^2\}\{(\xi+1)^2+[\zeta-2(n+1)]^2/\alpha^2\}}{\{(\xi-1)^2+(\zeta+2n)^2/\alpha^2\}\{(\xi-1)^2+[\zeta-2(n+1)]^2/\alpha^2\}}\right) \tag{46}$$

3. Discussion

The solutions for the line sources in a slab can be applied to calculate the temperature distribution in a floor operating as heat storage. The heat release from the floor to the room as function of time can be simulated. Thus it is possible to use the theory in dimensioning of the heat storage system and to simulate different time-dependent coil power and control strategies. Factors affecting the transient heat storage and release from floor to the room space are thickness, density, thermal conductivity and specific heat of floor and cover, distance of electric cables from each other and from floor surface, heat transfer coefficient including effects of convection and radiation, room air temperature, the power/length of cables, and the durations of on/off stages. Also the insulation under the floor and the temperature below are affecting. Instead of floor the heat storing slab could be a massive inner wall of a building. The model is also applicable in this case with space heating on both sides.

In the dimensioning the allowable temperature on the floor above the cover is an important dimensioning factor, since it affects comfort. Walking with bare feet should be comfortable. The temperature on the cover surface depends on the coil power and on the thermal properties of

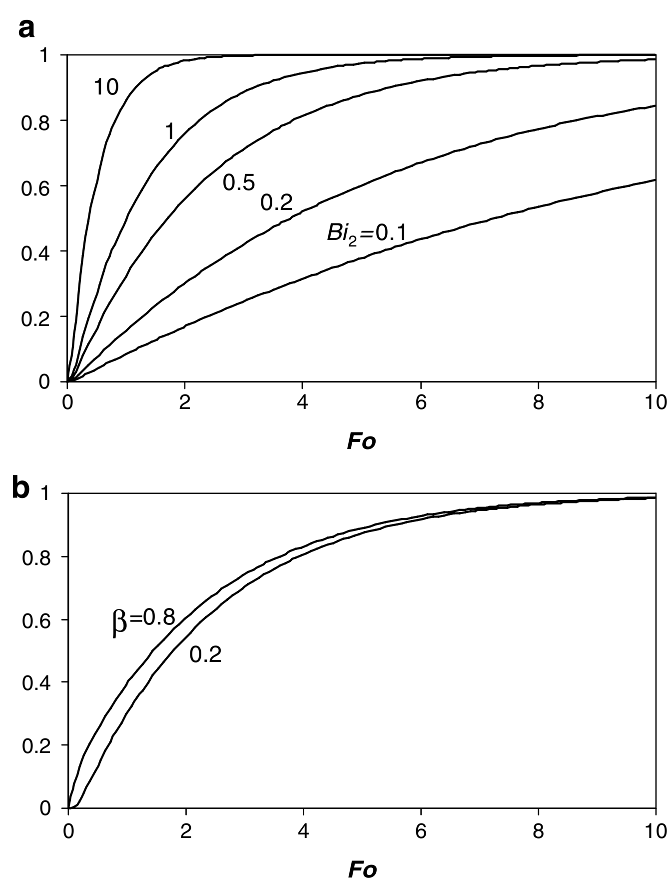

Fig. 3. Heat release/maximum release from floor as function of time after the power is switched on. (a) Effect of Bi_2 ($\beta = 0.4$, $Bi_1 = 0$). (b) Effect of β (location of coils) as the parameter ($Bi_1 = 0$, $Bi_2 = 0.5$).

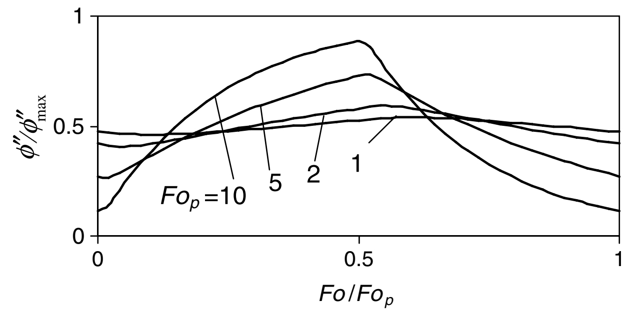

Fig. 4. Heat release from floor in cyclic operation ($Bi_1 = 0$, $Bi_2 = 0.5$, $\beta = 0.2$).

the floor and the cover and can be estimated by the model. A possible carpet on the floor cover affects the heat release.

The heat release from the floor surface after the coil power is switched on is shown in Fig. 3. The thickness of the slab R, which is included in $Fo = at/R^2$ (and also in $Bi = hR/\lambda$) has a significant effect on the heat storage capacity and on the time lag for the heat release. It is seen that also Bi has a relatively great effect on the heat release whereas the location of the cables (β) has not much effect.

Heat release from the floor as function of time in cyclic operation, where on and off periods are of equal durations, is illustrated in Fig. 4. The heat release will be more even with increasing thickness R (Fo_p decreases). Also decreasing Bi (by decreasing G''_2 using isolative cover on the floor) will make the heat release more even. The required total power for space heating, for example approximately constant heating power when outdoor temperature is constant, is obtained by using auxiliary heaters. The need for auxiliary heating can be calculated as the difference between the total heat demand and heat release from the floor.

The calculated temperature distribution in the floor in cyclic operation is shown in Figs. 5–9. In the calculations the following thermal and other properties were applied: $c_s = 880\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $\lambda_s = 1.37\ \mathrm{W\,m^{-1}\,K^{-1}}$, $\rho_s = 2300\ \mathrm{kg\,m^{-3}}$, $s_b = 3$ mm, $\lambda_b = 1.0\ \mathrm{W\,m^{-1}\,K^{-1}}$, $m \approx 0\ \mathrm{m^2\,K\,W^{-1}}$, $G''_1 = 0.2\ \mathrm{W\,m^{-2}\,K^{-1}}$, $T_1 = 5$ °C, $T_2 = 20$ °C, $h = 8\ \mathrm{W\,m^{-2}\,K^{-1}}$ (average by combined convection and radiation), $R = d = 0.1$ m.

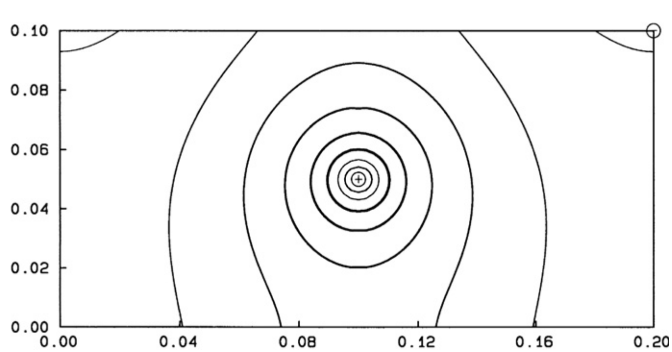

Fig. 5. Temperature distribution in the floor in cyclic heating at time $t = 2$ h after power has been switched on. Isotherms from 21.5 to 25.0 °C (step 0.5 °C). Maximum temperature is at location of line source denoted by symbol (⊕).

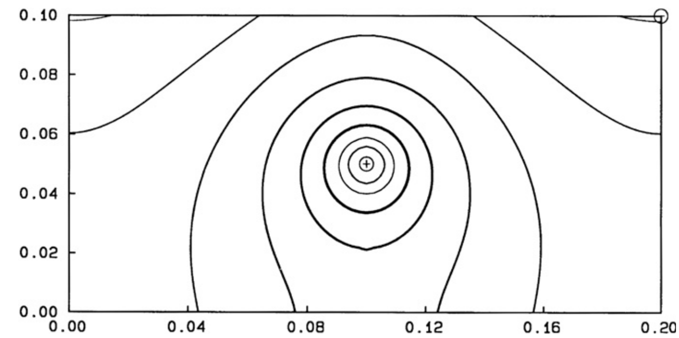

Fig. 6. Temperature distribution in the floor in cyclic heating at time $t = 6$ h after power has been switched on. Isotherms from 23.0 to 26.5 °C (step 0.5 °C). Maximum temperature is at location of line source denoted by symbol (⊕).

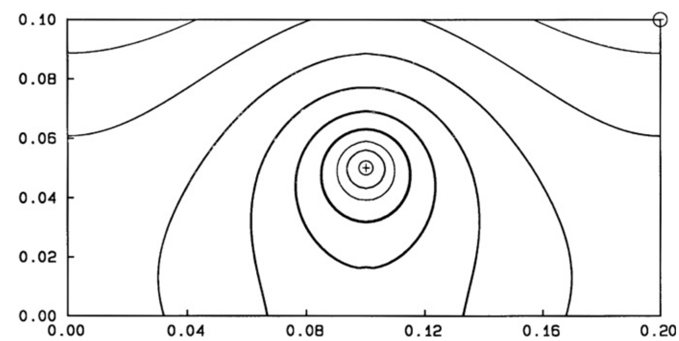

Fig. 7. Temperature distribution in the floor in cyclic heating at time $t = 12$ h after power has been switched on. Isotherms from 24.5 to 28.0 °C (step 0.5 °C). Maximum temperature is at location of line source denoted by symbol (⊕).

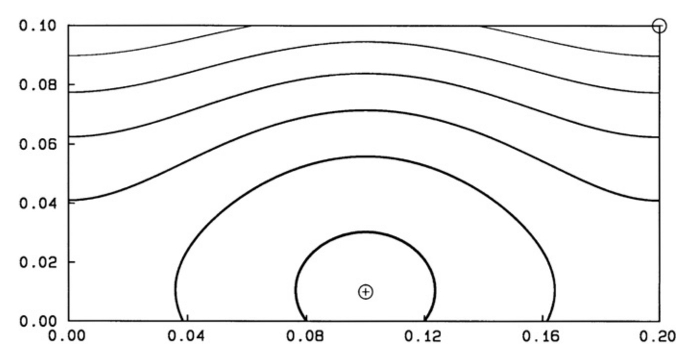

Fig. 8. Temperature distribution in the floor in cyclic heating at time $t = 12.5$ h (0.5 h after power has been switched off). Isotherms from 24.5 to 25.75 °C (step 0.25 °C). Maximum temperature is at location denoted by symbol (⊕).

From the thermodynamic point of view the use of electricity for heating is not efficient, since high grade energy is used at low temperature level. A floor with pipes and water circulation is an alternative way to store heat. Then it is possible to increase the storage capacity by connecting it to water heat storage. The benefit of a system with liquid circulation is that the heating system can more easily be

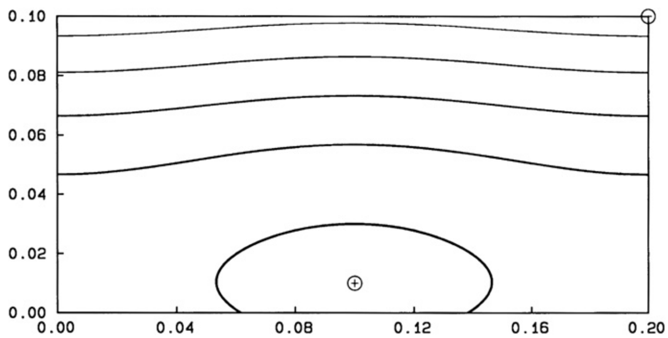

Fig. 9. Temperature distribution in the floor in cyclic heating at time $t = 13$ h (1 h after power has been switched off). Isotherms from 24.25 to 25.25 °C (step 0.25 °C). Maximum temperature is at location denoted by symbol (⊕).

changed later to use other sources of heat (district heating, heat from combustion, waste and solar heat). A liquid system can also be used for cooling and it might even be applied as a pre-heater of hot domestic water from 5 to 6 °C (obtained from municipal net in Finland) up to around 20 °C in summer.

The stationary temperature field generated by a single or several lines sources can be applied as an approximation of a temperature field generated by a pipe or pipes at constant outer wall temperature, since the isotherms in the vicinity and around a line source are almost circles. Here this is illustrated to calculate the heat losses from a single pipe in a semi-infinite solid (Fig. 10), but the method is applicable to several pipes in the ground and to a single or several pipes in a slab as well. The accuracy of this method can be reasoned from the question: how well does the shape of the isotherm describe the circular shape of the pipe. The stationary temperature field generated by the line a source is obtained by the numerical integration of Eq. (37). This

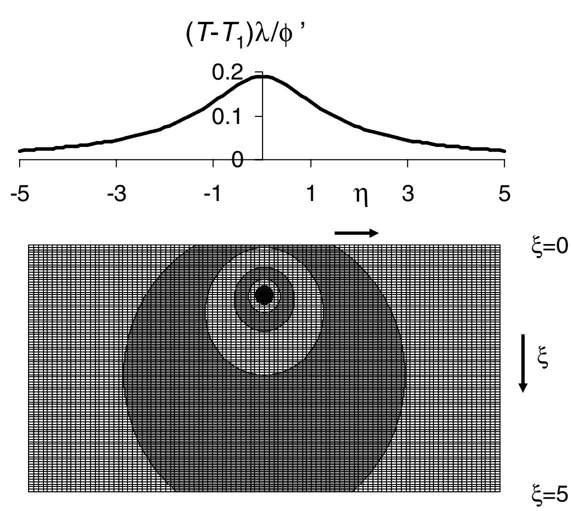

Fig. 10. Dimensionless surface temperature and temperature distribution $\Theta(\xi,\eta) = (T_s - T_1)\lambda_s/\phi'$ in a semi-infinite solid generated by a line source at point $\xi = 1$, $\eta = 0$. Black area $\Theta \geqslant 0.4$ and light and dark areas outwards are $\Theta = 0.4\ldots0.5$, $0.3\ldots0.4$, $0.2\ldots0.3$, $0.1\ldots0.2$ and $0\ldots0.1$, respectively.

dimensionless temperature field $\Theta = (T_s - T_1)\lambda/\phi'$ is shown as an example for the case $Bi = 1$ in Fig. 10. We consider a pipe with outer radius $R_p = \delta R$ and its centreline located at the line source. The approximate value of the field at the pipe outer surface can be calculated by integrating around the pipe surface

$$\bar{\Theta}_p = \frac{1}{2\pi}\int_0^{2\pi}\Theta_p\,d\varphi \approx [\Theta(1-\delta,0) + \Theta(1+\delta,0) + 2\Theta(1,\delta)]/4 \tag{47}$$

as average of four points at the pipe outer surface. Then we can calculate the heat loss from the pipe/length $\phi' \approx G'(T_p - T_1)$, where $G' = \lambda_s/\bar{\Theta}_p$. For example if $\delta = 0.2$, and $Bi = 1$, we get $\bar{\Theta} \approx [0.481 + 0.496 + 2 \times 0.482] = 0.485$. As it can be seen, the values at $\xi = 1 - \delta$ (0.481) and $\xi = 1 + \delta$ (0.496) are quite close. The accuracy is good, if the radius of the pipe is much smaller than its distance from the ground surface. If the radius of the pipe is large a more exact result for the heat loss would be obtained by adjusting the location of the line source from the centreline of the pipe $\xi = 1$ little towards the ground surface so that the temperatures at $\xi = 1 + \delta$ and $\xi = 1 - \delta$ become equal. This makes the boundary condition (constant temperature at outer surface of the pipe) even more exact. After G' is known the cooling of a fluid flowing in the pipe can be calculated by $T_f = T_1 + (T_{f0} - T_1)\exp[-G'z/(\dot{m}_f c_f)]$, if the heat conduction in the direction of the fluid flow (co-ordinate z) is assumed negligible.

The calculation process for a slab is similar. The average dimensionless temperature $\bar{\theta}_p$ is evaluated. We denote the stationary fields due to temperature difference and line sources, Eqs. (16) and (18), as $\theta_{a\infty} = \vartheta G(\xi)$ and $\theta_{e\infty} = \Phi H(\xi,\zeta)$. Then we evaluate the average value of the temperature field on the pipe surface $\bar{\theta}_p = \vartheta\bar{G} + \Phi\bar{H}$, where $\bar{G} = G(\beta)$ and $\bar{H} \approx [H(\beta+\delta,0) + H(\beta-\delta,0) + 2H(\beta,\delta)]/4$. Then the heat loss from the pipe surface is obtained from $\phi' \approx \lambda_s[T_p - T_1 - (T_2 - T_1)\bar{G}]/\bar{H}$. The accuracy can be improved in a similar way as before by adjusting the location of the line source so that temperatures at both sides of the pipe become equal.

4. Conclusions

Unsteady and steady thermal fields induced by a singe or several (infinite queue) of line sources in a slab or semi-infinite solid are derived using integral transform methods. Steady and intermittent heating with electric coils in a floor or a wall is considered as an application. Method to estimate thermal fields and heat losses from pipes in building structures, such as walls and floors and ground in steady state based on steady state line sources is discussed.

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